3. Complex Maps and Fractal Basin Boundaries

Until now, we have discussed one-dimensional maps and trajectories. The generalization to more dimensions is straightforward. For example, a two-dimensional map iterates (propagates) points in two-dimensional space, e.g.,

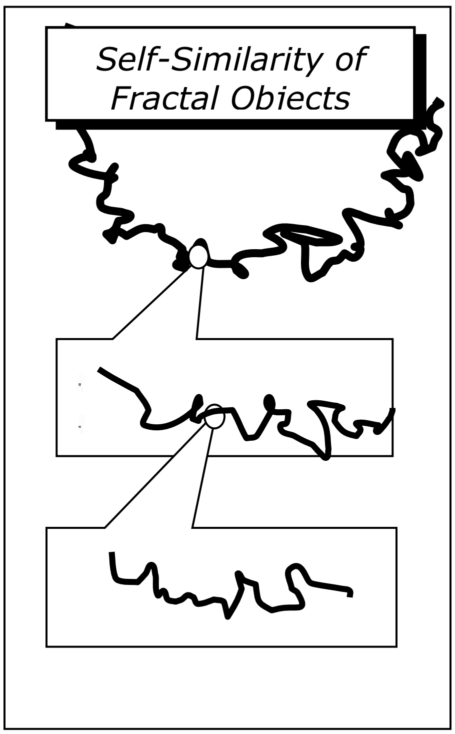
$$\begin{aligned} X_{i+1} &= f(X_i, Y_i) \qquad & Y_{i+1} &= g(X_i, Y_i) \\ f(X,Y) &= a + X^2 - Y^2; \quad & g(X,Y) &= b + 2X \cdot Y \end{aligned} \tag{17}$$

Catalytic chemical reaction dynamics discussed further below is an interesting example that can be represented by a multi-dimensional map.

As before, for an analysis of the dynamics of the system represented by such a map, it is interesting to inspect the ***space explored by the system trajectories***, as well as the ***parameter space*** (e.g., the gain factor μ for the logistic map) for which the system behaves orderly, settles down to a steady state, and for which it behaves chaotically. For example, one may wish to know, whether a trajectory started in a certain area of the coordinate space will converge to a certain attractor, approach a different attractor, or even escape to infinity.

Computer software is available to perform simulations of orderly and chaotic dynamics of various physical systems. The visual demonstrations in class have used the software package *MATHCAD* (C:\WINMCAD\Programs\Logistic_MAPN.MCD). It was observed that the trajectories were usually confined to a limited area around the attractors or stable (fix-) points. For example, for the logistic map (see Equ. 4), the values of $I_n = f(I_{n-1})$ always fluctuate within a smaller or larger sub-interval of [0,1] in intensity space, centered about the attractor/fixpoint. The intervals in parameter space corresponding to orderly and chaotic behavior are defined by the sign of the Liapunov exponent λ.

For multi-dimensional maps, for example the Lorenz weather map, the corresponding geometrical object is called ***basin of attraction***. These basins have a different form depending on the magnitude of the amplification parameter, the quantity μ in the case of the logistic map, and the attractor(s)/fix-point(s) considered.



If there is only one attractor and no possibility for the system to escape into infinity, the geometry of the associated basin of attraction is relatively simple. However, if there are ***more than one attractor*** and certain ***routes for escape***, a prediction of the behavior of the system becomes more difficult. The ability for reliable predictions depends on the extent to which the boundaries of the various basins are known. In fact, it may or ***may not be possible to know this boundary precisely***. For example, if this boundary is not a continuous, smooth curve but ragged and fragmented, there may be an unstable point always right next to one corresponding to a steady state of the system.

Such ragged curves are examples of fractal geometries. Continental coast lines, meandering rivers, the structure of dendrites, or the system of blood vessels in human organs are illustrations of fractal objects. These objects ***have no characteristic length scale***, they look basically similar, however closely one looks. This property is termed "***self-similar***". It turns out that the basin of attraction and the parameter space can have very intricate geometrical structure which replicates on each length scale. In the following, the most famous fractals, the ***Mandelbrot*** and ***Julia sets*** (of complex numbers) for the quadratic map will be discussed briefly.

The set of complex numbers $\{z = (x,y)\}$ defining the boundary of the basin of attraction of a (rational) complex map has a special name, ***Julia set*** (named after the mathematician *Julia* 1918). The boundaries of a similarly defined ***parameter space*** form another set of complex numbers, which for the map below is called ***Mandelbrot set***. Consider the simple quadratic map

$$z_{n+1} = f(z_n) = C + z_n^2 \tag{18}$$

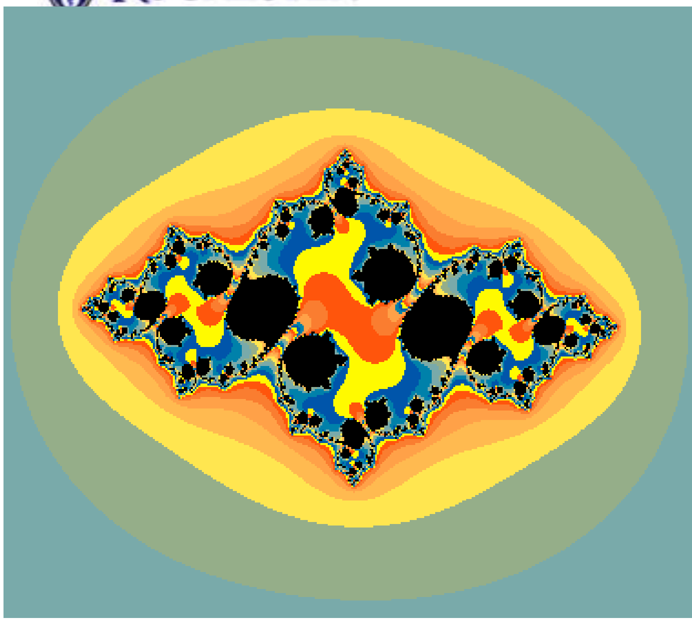
where *C* is a constant. Equ. (18) is an example of the structure of the map Equ. (17) and very similar to the logistic map. However, in this case, let the map be defined for all complex number.

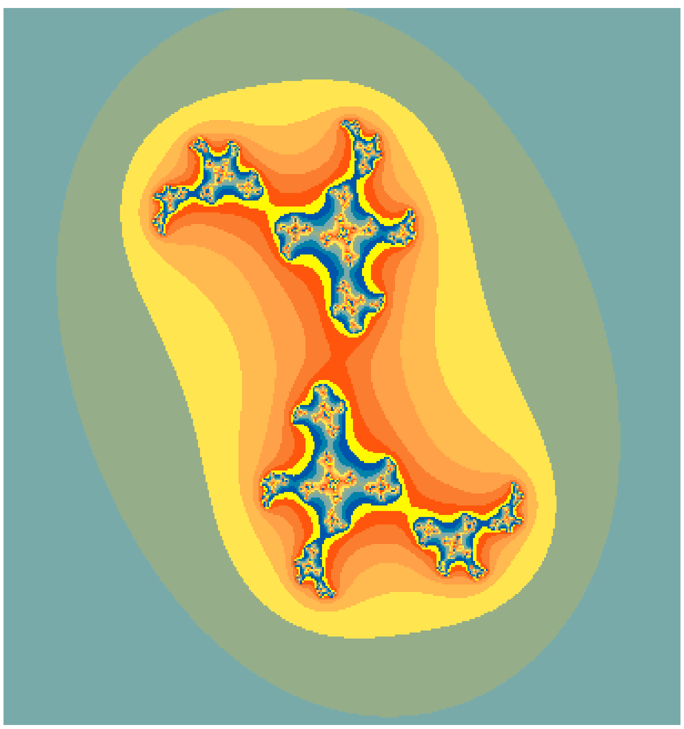
As before, one can find fixpoints, attractors, and repellors, which are now defined on the complex plane. This is, however, not of detailed interest in the present context of the geometry of the parameter space associated with a particular dynamic behavior described by this map. The question to be answered is, for what sets $\boldsymbol{J_c}$ of initial (complex) numbers *z* the trajectory remains bounded, i.e.,

$$J_c := \{z : |z_n| = |f^n(z)| < \infty\} \tag{19}$$

for a fixed parameter C. The set J_c is the ***Julia Set for the map f***. Two of these sets, belonging to two different parameters *C*, are shown on the figure below. The color scheme gives an indication of how fast a trajectory escapes to infinity, i.e., how many iteration it takes the map to go beyond a certain bound (circle) around its initial point. ***Julia sets are fractal objects***.

Given the potentially strong sensitivity to initial conditions of a system prone to chaotic behavior, a initial point *z*, the starting point of a trajectory can be called "***safe***" only, if it is surrounded by at least a ***small area, or volume, of points that all have the same character***. That is to say that of interest are those Julia sets of *f* that are connected, i.e., those that have a ***continuous interior***. The figure shows such a connected Julia set in the top panel, while the bottom set is a ***disconnected Julia set***. The





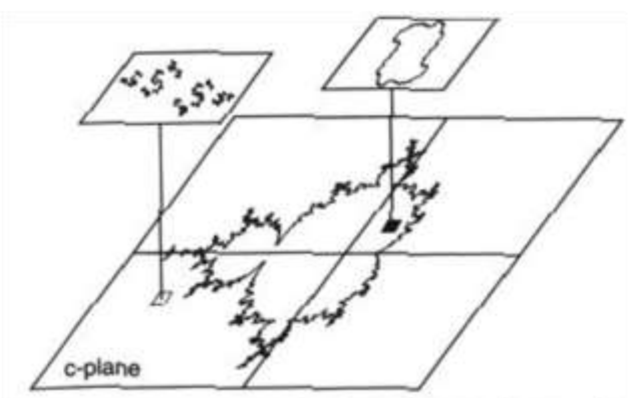
initial points z in this latter set cannot be considered "safe" in the sense defined above.

Whether a Julia set for a given map f is connected or disconnected depends on the values of the parameters determining that map. In the example of the quadratic map of Equ. (18), the parameters are the real and imaginary parts of the complex constant *C.*

The set of points *C* for which the Julia set J_c is connected is called the ***Mandelbrot set***. It turns out that these points form again a fractal object, one that has a main, connected body and a fractal boundary. As illustrated in the figure, points *C* outside this boundary result in disconnected Julia sets, point *C* inside the main body of the Mandelbrot set yield a connected ("safe") Julia set.

Decoding the escape behavior (rapidity of escape) in a color scheme, one obtains the interesting figure shown below in the top panel, produced with the *CHAOS* program {C:\CHAOS\CHAOS.BAT}.
The program allows one to "zoom" in on the smaller features. Doing that, one finds self-similarity, similarity of the picture on all length scales.

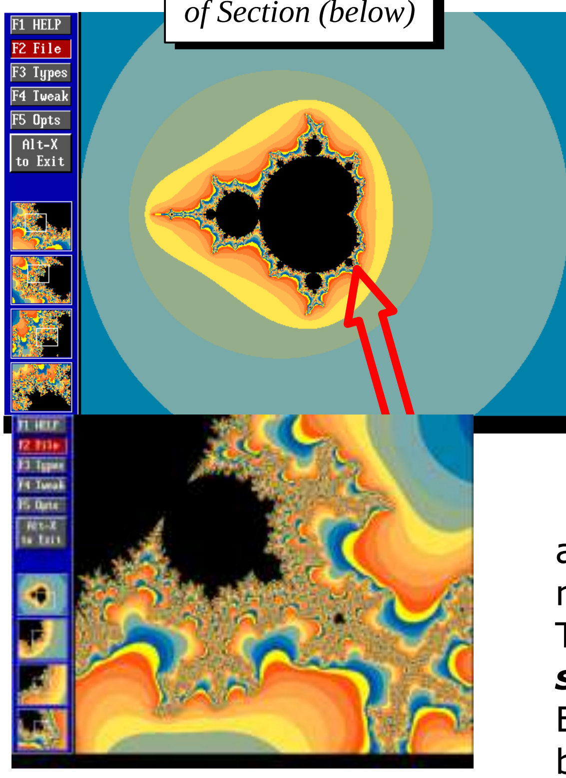
Relation between Julia and Mandelbrot Sets



It is also of interest to know for which parameter subspace the system trajectories remain in some basin of attraction and for which other parameter space, they would not remain confined anywhere, i.e., when they would escape into infinity.

Consider as an example, all trajectories starting from the initial point in the origin of the complex plane, $z_0 = 0$. Then, the interesting region of constants C is defined as the subspace on the ***complex plane*** (of constants C), for which trajectories starting at $z_0 = 0$ do not escape ***within a certain number of iterations*** (color-coded). The result of numerical calculations, which are not difficult but computation-intensive, are shown in the figure. Here, the boundary of the so-called ***Mandelbrot set*** is shown. Trajectories for C-values from the inside of the boundary are confined, those with constants C outside will escape into infinity.

Mandelbrot Set and Enlargement of Section (below)



The interesting observation regarding this ***fractal*** is that it looks the same on all length scales. The lower panel is a magnification of a small region of the figure in the upper panel. It is obviously ***self-similar***. In other words, the boundary between relative stability and instability of a dynamic system is not well defined. The dynamics depends extremely ***sensitively on initial conditions***. Enlarging a particular region on the boundary, e.g., at a bubble-like extrusion, shows again the same feature, here, another bubble-like extrusion. In a colored illustration, one can encode further properties of the trajectories, e.g., how many iterations it takes to leave a certain distance from the origin corresponds to different colors.

As already mentioned above, fractal structures are very common in nature. We find them in ice crystals, trees, and many other natural entities. The reason for the abundance of such structures is the underlying simplicity of the mechanisms that produce these structures. Again, structure and disorder are very close to each other. The following brief discussion of cellular automata illustrates this point in replication and self-organization.